# NON-EUCLIDEAN SPHERICS.

By DR. GEORGE BRUCE HALSTED.

As part of the general enlightenment springing from the creation of non-Euclidean geometry, every one now knows that spherical trigonometry is entirely independent of the parallel postulate. We wonder that anyone should have stopped to give proof of what is now so obvious.

Yet perhaps the most difficult article in all Lobachevski's "Geometrical Researches on the Theory of Parallels" is §35, which concludes, "Hence spherical trigonometry is not dependent upon whether in a rectilinear triangle the sum of the three angles is equal to two right angles or not."

Just so concludes Chapter XI of his New Elements: "Therefore the equations for spherical triangles remain the same whether we assume the angle of parallelism as constant or variable."

In §26 of Bolyai's Science Absolute of Space spherical trigonometry is established independently of the parallel postulate.

In his non-Euclidean space Bolyai found a uniform surface, $F$, whose proper geometry is Euclidean, its straight being $L$, the circle-limit. Lobachevski found the same, calling $F$ orisphere, $L$ oricycle.

But profound as was their genius it never questioned the assumption, of every three costraight points always one and only one is between the other two. As a consequence, the straight was for them of essence unclosed, and space infinite. So the characteristic geometry of the sphere was not given rank with that of the orisphere, and the non-Euclidean geometry of finite space remained unsuspected. Neither reached the conception that the totality of space may be finite.

The circle-limit was infinite and was conceived as the straight of the orisphere; the finite great circle was not conceived as the straight of the sphere. It remained for Riemann to perceive that the straight, though unbounded, need not be infinite, whence followed a new non-Euclidean geometry, now called by his name.

Beltrami showed that in Euclidean space there may be a surface a piece of which may perhaps represent a piece of the Bolyaian plane. Such is the surface of constant negative curvature, the pseudosphere.

True, it is impossible to represent the entire Bolyaian plane on a Beltrami surface without singular points; nevertheless, meaning by pseudospheres surfaces of revolution which have for meridians a tractrix or curve of equal tangents, we may say their characteristic geometry is Bolyaian.

Of late this doctrine has been filled out, completed by the beautiful

THEOREM OF BARBARIN: *Each of the three spaces, Euclidean, Bolyaian, Riemannean, contains surfaces of constant curvature of which the geodesic lines have the metric properties of the straights of the three spaces.*

These surfaces are (1) the tubes or surfaces equidistant from a straight,

it being possible for the distance to be infinite, which gives the orispheres, (characteristic geometry, Euclidean); (2) the pseudospheres (characteristic geometry, Bolyaian); (3) the spheres (characteristic geometry, Riemannean).

In 1879 Killing made clear the distinction between Riemannean space and what he then called its polar form, by Klein called simple elliptic space. This latter, Killing thinks, had been entirely missed by Riemann, as we know it was by Helmholtz even up to 1876 when he still reproduced the old but false theorem that in space of positive curvature two geodetic lines, if they in general intersect, must necessarily intersect in two points. Such a space of constant curvature Klein calls *spherical* in contradistinction to the simple elliptic in which the assumption, two points determine a straight, has no exception.

Killing it was also who first proved that besides the Euclidean, Bolyaian, simple elliptic, the spherical or old Riemannean is the only, the sole space which as a whole can be freely moved in itself. There are undertypes in abundance where the free mobility of figures only holds so long as the dimensions of the figures do not surpass a certain size; a series of topologically distinguishable spaces which for bounded (simply connected) parts are Euclidean, Bolyaian, simple elliptic. Moreover it has been demonstrated, so far as concerns the surfaces of constant positive curvature, to which the Riemannean geometry applies, that apart from the sphere there is no other closed surface of this sort. The sphere is the only closed surface of positive constant curvature without singularities.

All this intensifies the importance of surface spherics, two-dimensional spherics, pure spherics, intrinsic spherics, Riemannean spherics, double-elliptic spherics, non-Euclidean spherics.

Fortunately a place even in general education has been held open for this newcomer. All the theorems of the so-called "solid geometry" of the schools which relate solely to the surface of the sphere, there obtained by using the parallel postulate, by dragging in the globe which in Euclidean geometry is inside the sphere, in fact by considering the sphere as the covering belonging to such a globe, and therefore tri-dimensional, in a three-way-infinite manifold, are really theorems of that simpler finite manifold the sphere, having no dependent relation to the Euclidean straight, plane, or space. How obvious, then, that these theorems should be developed entirely from the assumptions which characterize the sphere.

Even what is ordinarily conceived of as the shape of the sphere is not wholly irrelevant, for, using the terminology of our Euclidean intuition, if the surface or covering of a globe be detached from the globe, any surface into which it can be bent without stretching, into which it can be flexed, is a double elliptic surface, a surface of constant positive curvature with its proper geometry, and what is ordinarily thought of as the free mobility of figures in it may remain; but somewhere on this surface has come a singularity, it is no longer free from singularities.

As an illustration in still lower terms of the meaning of intrinsic properties of the sphere, take the circle, the closed curve which will slide on its trace,

which is mobile in itself as a whole. This occurs in the Euclidean, Bolyaian, Riemannean plane with its intrinsic properties unchanged; but consider its radius, and it flies apart into three. The circumference of a circle in the Euclidean plane equals $2\pi r$; in the sphere the circumference is less than $2\pi r$; in the pseudo-sphere the circumference is greater than $2\pi r$. The intrinsic properties of the sphere are just what we want. Since they are utterly independent of the parallel-postulate, the simplest spherics must be non-Euclidean.

The student's familiarity with the sphere under its old Euclidean aspect as glove to a globe is also an advantage, which to all needing an introduction to the new ways of treating geometries decides in favor of intrinsic spherics as against simple elliptic planimetry with its unilateral plane which we can so strangely get through without going through. In spherics we have familiar material to present in the new light, with the new methods, to be therein acquired that they may be then retained for analogous conquest of unfamiliar realms.

When instead of building up theorems on polyhedral angles and then cutting them back into spherics, we realize the more complex theorems of angloids as already given in the simpler points of the sphere, we appreciate the practical in the theoretic.

How important, how enlightening to set forth the fundamental assumptions which give by pure logic all the relations of spheric figures, and to see developed therefrom the familiar system of theorems which so long constituted spherical geometry.

The old straight line, the old great circle are dissipated, volatilized, and in their place comes the *straightest* to which now applies the old definition of the straight line, "a line which pierces space evenly, so that a piece of space from along one side of it will fit any side of any other portion."

In vulgar phrase, the straightest turns neither to the right nor to the left so far as the sphere is concerned. But motion can never be fundamental, and it is the assumptions which really make the space. There is one geometric entity back even of the straightest, the point. It is the relation of the straightest to the point which differentiates the spheric from the simple elliptic. "The straight," says Mansion, "is a line determined by any two of its points, sufficiently near." But what is the meaning of 'sufficiently near'?

As long ago as 1877, I overcame these difficulties by building up the system of spherical geometry on a set of assumptions expressing only the few fundamental relations of points and straightests. Clearness is subserved by using 'straightest' as the designation for the spheric straight.

Line is a word which has always been used for the genus of which curve is a species, and of late such distractingly, bewilderingly complex curves and lines have appeared, that line in general should have no longer a place in the elements. Point and straightest are consciously accepted as elements to which specific assumptions give the requisite precision. To forestall controversy, one may reserve the word definition to mean an agreement to substitute a simple term or symbol for more complex terms or symbols.

Instead then of Mansion's definition we have what we prefer to call an Assumption of Association: I 1. For every point of the sphere there is always one and only one other point which with the first does not determine a straightest.

This second point we will call the *opposite* of the first. After three more assumptions, we come to a very fundamental yet complex question, the arrangement of one sort of element on the other. The word 'order' is so common that we are not conscious of its complexity.

What is the difference between $AB$ and $BA$? Does it not involve the assumption of a third, perhaps a fourth something? Of two sounds in time can one come first and the other afterwards unless we assume also the idea of a past? Could there be a present and a future without the idea of a past? To get a future must not the present act as past? Is there any difference between the point-pair $AB$ and the point-pair $BA$ apart from their relation to a bearer, a carrier, be this merely time itself? There being no elements but points, and only three of these, can there be a relation among these three points called 'order'?

If the points $ABC$ and no others are on a bearer, they may be said to have order or no order according as the bearer is open or closed. If they have order it may be $ABC$ with $CBA$, or $ACB$ with $BCA$, or $BAC$ with $CAB$, according to the bearer.

There exists a particular geometry, purely qualitative, making no use of the notion of straightness or planeness, but instead only of those of line and surface. This is the so-called *analysis sitûs*. Yet here remains order. Is it not then impossible that without loss of generality order should be subjected to straightness? For simplicity then, for certainty, for accuracy and ease, let us come down from the general idea of order to a more specific idea which we intend to apply only to points on a straight or a straightest, and for which we will take as available the unused word 'betweenness.' Hilbert in 1899 stressed the importance of *between* for the arrangement of costraight points. His treatment was extraordinarily simplified by the elegant proof* of my pupil R. L. Moore that one of his assumptions was redundant. But the problem for points on a straightest is far more difficult. Three terms cannot have a cyclic order, and to say of three points on a circle, that each is between the other two is to waste 'between'. So is the inexpert remark that a point, though it does not divide a straightest into two sects, yet makes of it a single piece in which the points are arranged in a natural order; that is, the words "follow," "precede," "lie between" are applicable.

The great working value of betweenness, is that when a point is known to be between two points, it is thereby located on one particular given straightest. But if we accept 'between' in the above inexpert sense, then to say $B$ is between $A$ and $C$ may mean absolutely nothing, since if $A$ and $C$ are opposite, every other point of the sphere is on a straightest with them and, in the inexpert sense, between them. As a consequence, therefore, the very first of our Assumptions

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*MONTHLY, Vol. IX, April, 1902, pp. 100-1. Cf. E. H. Moore, *Transactions*, Vol. 3 (1902), pp. 142-158.

of betweenness on the sphere, to specify how "between" is to be used of points in a straightest on a sphere, must be:

II 1. No point is between two opposites. This is reinforced by

II 2. Between any two points not opposites there is always a third point.

Filling out this scheme, we have a 'between' that can be used, for example in the definition: Two points $A$ and $B$, not opposites, upon a straightest $a$, we call a *sect*, and designate it with $AB$ or $BA$. The points between $A$ and $B$ are said to be points of the sect $AB$ or also situated *within* the sect $AB$. The remaining points of the straightest $a$ are said to be situated without the sect $AB$. The points $A$, $B$ are called *end-points* of the sect $AB$.

The next set of assumptions are of congruence. This is a matter which the inexpert suppose they can finish briefly thus:

Definition. Geometrical figures which may be carried over into one another by rigid motions are said to be congruent ($\equiv$).

Theorem. A sect is congruent to itself reversed in direction.

Proof. The point $A$ may be be applied to the point $B$ and the direction $AB$ to the direction $BA$. Then $B$ will fall upon $A$: for otherwise the part and the whole would be congruent.

But this treatment is entirely inadmissible.

To define congruence by rigid motion is false and fallacious, since the intuition of rigid motion involves, contains, and uses the congruence idea. We must base the idea of motion on the congruence idea.

A man of whom it has been said: "He was by far the most eminent Amerisan of the Colonial Period, whether we regard the influence of his labors and opinions, upon his own time, in his own country, their wide diffusion in others, or that survival of prestige and authority which yet perpetuates his name and memory," Jonathan Edwards, who died president of Princeton, says, "Motion is a body's existing successively in all the immediate contiguous parts of any distance, without continuing for any time in any one of them."

Its geometric substratum, then, is the preexistence of a series of congruent figures. So rigid motion presupposes congruence.

Moreover, we do not need the troublesome idea 'direction.' In the plane, 'same direction' assumes the whole theory of parallels. On the sphere no two straightests have the same direction, since no two are parallel, yet every two have the same direction, since they go from the same point to the same point. Nor is anything gained by agreeing to call a ray a direction. So preceding motion must come congruence, the idea to be made precise by assumptions.

But right here an unexpected and hitherto unsuspected simplification is possible. In his first congruence axiom, III 1, Hilbert explicitly assumes "Every sect is congruent to itself, that is, always $AB \equiv AB$." Now this assumption is redundant, as was Hilbert's II 4. It is a demonstrable theorem. Two proofs of it have been given me by R. L. Moore, based on the Assumptions of congruence:

III 1. If $A$ not $\equiv B$, and $A'$ not $\equiv C'$, then on ray $A'C'$ there is one and only one point, $B'$, such that $AB \equiv A'B'$.

III 2. If $AB \equiv A'B'$ and $A'B' \equiv A''B''$, then $AB \equiv A''B''$.

III 3. If $B$ is between $A$ and $C$ and $B'$ is between $A'$ and $C'$, and $AB \equiv A'B'$ $BC \equiv B'C'$, then $AC \equiv A'C'$.

We next define angle as two rays from a common point, and assume as

III 4: On a given side of a given ray there is one and only one angle congruent to a given angle.

Instead now of two congruence assumptions

III *a*: Angles congruent to the same are congruent to each other, and

III *b*: Euclid I. 4,

we may prove these two as theorems by defining congruent angles in terms of congruent sects, and assuming

III 5: Euclid I. 8: If $A$, $B$, $C$ are non-costraight and so are $A'$, $B'$, $C'$, and $C$ is between $B$ and $D$, and $C'$ is between $B'$ and $D'$, and $AB \equiv A'B'$, $BC \equiv B'C'$, $CA \equiv C'A'$ and $BD \equiv B'D'$, then $AD \equiv A'D'$.

In itself a point-pair not only has no order, it does not even possess sense. But a sect, a point-pair on a straightest, has sense, and $AB \equiv BA$ must be proved or assumed. It may be assumed without introducing any divergence between the old concept, superposable, and the more fundamental concept, congruent. $AB$ is superposed on $BA$ by a semi-rotation about their common mid-point. On the sphere an angle, the figure two rays from the same initial point, has sense.

The analogue of the semi-rotation of a sect about its mid-point is the semi-rotation of an angle about its mid-ray.

If two figures have central symmetry in a plane, either can be made to coincide with the other by turning it in the plane through two right angles. This holds good when for "plane" we substitute "sphere." Any sect and its inverse, $AB$ and $BA$, are such figures. They are symcentral about the mid-point of either.

If two figures have axial symmetry in a plane, they can be made to coincide by folding the plane over along the axis, but not by any sliding in their plane. Two axially symmetrical figures in a plane can be brought into coincidence by a semi-revolution of one about the axis. That is, we must use the third dimension of space, and then their congruence depends on the property of the plane that its two sides are indistinguishably alike, any plane will completely fit its trace after being turned over. This procedure, folding over along a line, can have no place in a strictly two-dimensional geometry.

So figures with axial symmetry on a sphere cannot be made to coincide. Let symmetric henceforth mean axially symmetric, and be denoted by $\cdot|\cdot$. A spherical angle and its inverse, $\widehat{\sphericalangle}(h, k)$ and $\widehat{\sphericalangle}(k, h)$ are not symcentral and cannot be brought into coincidence. Should we take as a definition:

An angle is called symmetric to another to whose inverse it is congruent; $\sphericalangle(h, k) \cdot|\cdot \sphericalangle(v, w)$ when $\sphericalangle(h, k) \equiv \sphericalangle(w, v)$, then from $\sphericalangle(h, k) \equiv \sphericalangle(h, k)$ comes $\sphericalangle(h, k) \cdot|\cdot \sphericalangle(k, h)$, but these two are not superposable, cannot be made to coincide.

To those then who have made ideal superposition the basis and test of con-

gruence, the fact that a spherical angle can in no way be placed upon its inverse introduces a radical difference between their presentation of spherics and the familiar presentations of plane geometry. For them many theorems would be bifurcate, for example, Theorem: Any two right angles are either congruent or symmetric. But since congruence in no way depends on the subsequenr concept, motion, nothing is more simple than to assume $\widehat{\sphericalangle}(h, k) \equiv \widehat{\sphericalangle}(k, h)$.

The three points which determine a triangle have no order and no betweenness, in the specific meaning, for that would make two of a like character and the third of a different character. But as a triad of sects, a triangle has as an individual 'umlaufssinn', tour-sense.

If the congruence of sects is connected with that of angles by a triangle assumption, then if this is restricted to triangles of the same tour-sense, it will not give us the congruence of the basal angles in an isosceles triangle. To attain this is in that case needful the adjunction of one or two continuity assumptions.

Thus we see that beyond the one dimensional space-symmetry implied in the congruence of sect and angle with their inverses, there is a quite distinct two-dimensional space symmetry. This is usually unrecognized in the ordinary treatment of plane geometry, since a plane triangle can have its tour-sense changed by turning it over in the third dimension.

In two-dimensional spherics this is impossible, so although this two-dimensional space-symmetry is presumed in the triangle-assumption's non-recognition of tour-sense, yet it is customary to note its perception in the distinction between congruent and symmetric triangles.

Had this distinction been retained further back, namely for angles, we might have used it in the definition: Two triangles are called symmetric when their corresponding sides are congruent and their corresponding angles are symmetric. But a definition perhaps more desirable comes from setting up the distinction of right side and left side of an angle.

Besides the assumptions of congruence, we need no metric assumptions, and as definition or axiom are well rid of that snaky phrase: "A straight line is the shortest distance between two points." In this reference see Georg Hamel: Ueber die Geometrieen, in denen die Geraden die Kürzesten sind, Math. Ann. Bd. 57, 1903.

Now as to continuity, there seems more call for such assumptions in spherics than in planimetry, since, for example, in the plane a sect may be readily divided into any desired number of parts, while in spherics recourse must be had to a continuity assumption even to show that a given sect has third parts, that one-third of a given sect exists. Nevertheless continuity remains costly. Even Hilbert has not succeeded in attaining a simple treatment of it. His 'Axiom der Vollständigkeit' and 'Axiom der Nachbarschaft' seem inelegant lumps in his beautiful and fine mosaic. Russell says (*Principles of Mathematics*, p. 440): "Whether the axiom of continuity be true as regards our actual space is a question which I see no means of deciding. "For any such question must be empirical, and it would be quite impossible to distinguish empirically what may be

called a rational space from a continuous space."

In my *Rational Geometry* I treat Non-Euclidean Spherics without any continuity assumption whatsoever.

## PROBLEMS FOR SOLUTION.

### ALGEBRA.

E. Kesner, Salida, Col., solved 208 and 209.

**211. Proposed by G. W. GREENWOOD, M. A. (Oxon), Lebanon, Ill.**

Prove that $p-qx$ and $q-px$ tend to equality as $x$ dimishes to zero, but yet that their limits are not equal. [Edwards' *Differential Calculus*, p. 7, ex. 10.]

**Solution by M. E. GRABER, A. M., Heidelberg University, Tiffin, Ohio.**

The limits of $p-qx$ and $q-px$ as $x$ dimishes to zero, are $p$ and $q$. Suppose $p>q$, then by subtraction the difference at the limit is $p-q$ and the difference for any value of $x$ is $p-q-qx+px$ which evidently decreases as $x$ diminishes. Likewise for $q>p$. Therefore $p-qx$ and $q-px$ tend to equality as $x$ diminishes to zero.

Also solved by G. B. M. Zerr, and J. Scheffer.

**212. Proposed by F. P. MATZ, Ph. D., Sc. D.**

Solve $\dfrac{x^2+2x+3}{x^2-2x+3}+\dfrac{x^2-2x+3}{x^2+2x+3}=\dfrac{10}{3}$.

**I. Solution by O. S. WESTCOTT, Waller High School, Chicago, Ill., ELMER SCHUYLER, Brooklyn, N. Y., and J. SCHEFFER, Hagerstown, Md.**

If we write $\dfrac{x^2+2x+3}{x^2-2x+3}=y$, then will $y+\dfrac{1}{y}=\frac{10}{3}$, and $y=3$ or $\frac{1}{3}$.

*a*) If $\dfrac{x^2+2x+3}{x^2-2x+3}=3$, $x=3$ or 1.

*b*) If $\dfrac{x^2+2x+3}{x^2-2x+3}=\frac{1}{3}$, $x=-3$ or $-1$.

**II. Solution by A. H. HOLMES, Brunswick, Maine.**

Clearing the given equation of fractions and reducing, we obtain $x^4-10x^2=-9$, whence $x=1$, $-1$, 3, $-3$.

Also solved by G. W. Greenwood. E. L. Rich, M. E. Graber, G. B. M. Zerr, F. D. Posey, S. S. Flory, L. E. Newcomb, and E. Kesner.

**213. Proposed by F. P. MATZ, Ph. D., Sc. D.**

Find the two roots of the equation $x^5-209x+56=0$, whose product is unity.